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## FLAT MANIFOLDS WITH HOLONOMY GROUP $\mathbb{Z}_2^K$ OF DIAGONAL TYPE

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### Abstract

We consider relations between two families of flat manifolds with holonomy group  $\mathbb{Z}_2^K$  of diagonal type: the family  $\mathcal{RBM}$  of real Bott manifolds and the family  $\mathcal{GHW}$  of generalized Hantzsche–Wendt manifolds. In particular, we prove that the intersection  $\mathcal{GHW} \cap \mathcal{RBM}$  is not empty. Moreover, we consider some class of real Bott manifolds without  $\text{Spin}$  and  $\text{Spin}^C$  structure. There are given conditions (Theorem 1) for the existence of such structures. As an application a list of all 5-dimensional oriented real Bott manifolds without  $\text{Spin}$  structure is given.

### 1. Introduction

Let  $M^n$  be a flat manifold of dimension  $n$ . By definition, this is a compact connected, Riemannian manifold without boundary with sectional curvature equal to zero. From the theorems of Bieberbach ([2]) the fundamental group  $\pi_1(M^n) = \Gamma$  determines a short exact sequence:

$$(1) \quad 0 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \xrightarrow{p} G \rightarrow 0,$$

where  $\mathbb{Z}^n$  is a torsion free abelian group of rank  $n$  and  $G$  is a finite group which is isomorphic to the holonomy group of  $M^n$ . The universal covering of  $M^n$  is the Euclidean space  $\mathbb{R}^n$  and hence  $\Gamma$  is isomorphic to a discrete cocompact subgroup of the isometry group  $\text{Isom}(\mathbb{R}^n) = O(n) \times \mathbb{R}^n = E(n)$ . Conversely, given a short exact sequence of the form (1), it is known that the group  $\Gamma$  is (isomorphic to) the fundamental group of a flat manifold if and only if  $\Gamma$  is torsion free. In this case  $\Gamma$  is called a Bieberbach group. We can define a holonomy representation  $\phi: G \rightarrow GL(n, \mathbb{Z})$  by the formula:

$$(2) \quad \forall g \in G, \quad \phi(g)(e_i) = \tilde{g}e_i(\tilde{g})^{-1},$$

where  $e_i \in \Gamma$  are generators of  $\mathbb{Z}^n$  for  $i = 1, 2, \dots, n$ , and  $\tilde{g} \in \Gamma$  such that  $p(\tilde{g}) = g$ .

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In this article we shall consider only the case

$$(3) \quad G = \mathbb{Z}_2^k, \quad 1 \leq k \leq n-1, \quad \text{with} \quad \phi(\mathbb{Z}_2^k) \subset D \subset GL(n, \mathbb{Z}),$$

where  $D$  is the group of all diagonal matrices. We want to consider relations between two families of flat manifolds with the above property (3): the family  $\mathcal{RBM}$  of real Bott manifolds and the family  $\mathcal{GHW}$  of generalized Hantzsche–Wendt manifolds. In particular, we shall prove (Proposition 1) that the intersection  $\mathcal{GHW} \cap \mathcal{RBM}$  is not empty.

In the next section we consider some class of real Bott manifolds without  $\text{Spin}$  and  $\text{Spin}^\mathbb{C}$  structure. There are given conditions (Theorem 1) for the existence of such structures. As an application a list of all 5-dimensional oriented real Bott manifolds without  $\text{Spin}$  structure is given, see Example 2. In this case we generalize the results of L. Auslander and R.H. Szczarba, [1] from 1962, cf. Remark 1. At the end we formulate a question about cohomological rigidity of  $\mathcal{GHW}$  manifolds.

## 2. Families

**2.1. Generalized Hantzsche–Wendt manifolds.** We start with the definition of generalized Hantzsche–Wendt manifold.

**DEFINITION 1** ([17, Definition]). A generalized Hantzsche–Wendt manifold (for short  $\mathcal{GHW}$ -manifold) is a flat manifold of dimension  $n$  with holonomy group  $(\mathbb{Z}_2)^{n-1}$ .

Let  $M^n \in \mathcal{GHW}$ . In [17, Theorem 3.1] it is proved that the holonomy representation (2) of  $\pi_1(M^n)$  satisfies (3). The simple and unique example of an oriented 3-dimensional generalized Hantzsche–Wendt manifold is a flat manifold which was considered for the first time by W. Hantzsche and H. Wendt in 1934, [9]. Let  $M^n \in \mathcal{GHW}$  be an oriented,  $n$ -dimensional manifold (a HW-manifold for short). In 1982, see [17], the second author proved that for odd  $n \geq 3$  and for all  $i$ ,  $H^i(M^n, \mathbb{Q}) \cong H^i(\mathbb{S}^n, \mathbb{Q})$ , where  $\mathbb{Q}$  are the rational numbers,  $\mathbb{S}^n$  denotes the  $n$ -dimensional sphere and “ $\cong$ ” denotes an isomorphism of groups. Moreover, for  $n \geq 5$  the commutator subgroup of the fundamental group  $\pi_1(M^n) = \Gamma$  is equal to the translation subgroup  $([\Gamma, \Gamma] = \Gamma \cap \mathbb{R}^n)$ , [16]. The number  $\Phi(n)$  of affine non equivalent HW-manifolds of dimension  $n$  grows exponentially, see [14, Theorem 2.8], and for  $m \geq 7$  there exist many isospectral manifolds non pairwise homeomorphic, [14, Corollary 3.6]. The manifolds have an interesting connection with Fibonacci groups [18] and the theory of quadratic forms over the field  $\mathbb{F}_2$ , [19]. HW-manifolds have no  $\text{Spin}$ -structure, [13, Example 4.6 on p. 4593].

The (co)homology groups and cohomology rings with coefficients in  $\mathbb{Z}$  or  $\mathbb{Z}_2$ , of generalized Hantzsche–Wendt manifolds are still not known, see [5] and [6].

We finish this overview with an example of generalized Hantzsche–Wendt manifolds which have been known already in 1974.

EXAMPLE 1. Let  $M^n = \mathbb{R}^n / \Gamma_n$ ,  $n \geq 2$  be manifolds defined in [12] (see also [17, p. 1059]), where  $\Gamma_n \subset E(n)$  is generated by  $\gamma_0 = (I = \text{id}, (1, 0, \dots, 0))$  and

$$(4) \quad \gamma_i = \left( \begin{bmatrix} 1 & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \frac{1}{2} \\ \vdots \\ 0 \end{pmatrix} \right) \in E(n),$$

where  $-1$  is in the  $(i, i)$  position and  $1/2$  is the  $(i + 1)$  coordinate of the column,  $i = 1, 2, \dots, n - 1$ .  $\Gamma_2$  is the fundamental group of the Klein bottle.

**2.2. Real Bott manifolds.** We follow [3], [11] and [15]. To define the second family let us introduce a sequence of  $\mathbb{R}P^1$ -bundles

$$(5) \quad M_n \xrightarrow{\mathbb{R}P^1} M_{n-1} \xrightarrow{\mathbb{R}P^1} \dots \xrightarrow{\mathbb{R}P^1} M_1 \xrightarrow{\mathbb{R}P^1} M_0 = \{\text{a point}\}$$

such that  $M_i \rightarrow M_{i-1}$  for  $i = 1, 2, \dots, n$  is the projective bundle of a Whitney sum of a real line bundle  $L_{i-1}$  and the trivial line bundle over  $M_{i-1}$ . We call the sequence (5) a *real Bott tower* of height  $n$ , [3].

DEFINITION 2 ([11]). The top manifold  $M_n$  of a real Bott tower (5) is called a real Bott manifold.

Let  $\gamma_i$  be the canonical line bundle over  $M_i$  and set  $x_i = w_1(\gamma_i)$ . Since  $H^1(M_{i-1}, \mathbb{Z}_2)$  is additively generated by  $x_1, x_2, \dots, x_{i-1}$  and  $L_{i-1}$  is a line bundle over  $M_{i-1}$ , one can uniquely write

$$(6) \quad w_1(L_{i-1}) = \sum_{k=1}^{i-1} a_{k,i} x_k$$

with  $a_{k,i} \in \mathbb{Z}_2 = \{0, 1\}$  and  $i = 2, 3, \dots, n$ .

From above  $A = [a_{ki}]$  is a strictly upper triangular matrix<sup>1</sup> of size  $n$  whose diagonal entries are 0 and other entries are either 0 or 1. Summing up, we can say that the tower (5) is completely determined by the matrix  $A$ . From [11, Lemma 3.1] we can

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<sup>1</sup> $a_{k,i} = 0$  for  $k \geq i$ .

consider any real Bott manifold  $M(A)$  in the following way. Let  $M(A) = \mathbb{R}^n / \Gamma(A)$ , where  $\Gamma(A) \subset E(n)$  is generated by elements

$$(7) \quad s_i = \left( \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & (-1)^{a_{i,i+1}} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & (-1)^{a_{i,n}} \end{bmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{2} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \right) \in E(n),$$

where  $(-1)^{a_{i,i+1}}$  is in the  $(i+1, i+1)$  position and  $1/2$  is the  $(i)$  coordinate of the column,  $i = 1, 2, \dots, n-1$ .  $s_n = (I, (0, 0, \dots, 0, 1)) \in E(n)$ . From [11, Lemma 3.2, 3.3]  $s_1^2, s_2^2, \dots, s_n^2$  commute with each other and generate a free abelian subgroup  $\mathbb{Z}^n$ . It is easy to see that it is not always a maximal abelian subgroup of  $\Gamma(A)$ . Moreover, we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & \Gamma(A) & \longrightarrow & \mathbb{Z}_2^k \longrightarrow 0 \\ & & \uparrow i & & \parallel & & \uparrow p \\ 0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \Gamma(A) & \longrightarrow & \mathbb{Z}_2^n \longrightarrow 0 \end{array}$$

where  $k = \text{rank}_{\mathbb{Z}_2}(A)$ ,  $N$  is the maximal abelian subgroup of  $\Gamma(A)$ , and  $p: \Gamma(A)/\mathbb{Z}^n \rightarrow \Gamma(A)/N$  is a surjection induced by the inclusion  $i: \mathbb{Z}^n \rightarrow N$ . Here  $\text{rank}_{\mathbb{Z}_2}(A)$  denotes a rank of the matrix  $A$ . From the first Bieberbach theorem, see [2],  $N$  is a subgroup of all translations of  $\Gamma(A)$  i.e.  $N = \Gamma(A) \cap \mathbb{R}^n = \Gamma(A) \cap \{(I, a) \in E(n) \mid a \in \mathbb{R}^n\}$ .

DEFINITION 3 ([3]). A binary square matrix  $A$  is a Bott matrix if  $A = PBP^{-1}$  for a permutation matrix  $P$  and a strictly upper triangular binary matrix  $B$ .

Let  $\mathcal{B}(n)$  be the set of Bott matrices of size  $n$ .<sup>2</sup> Since two different upper triangular matrices  $A$  and  $B$  may produce (affinely) diffeomorphic ( $\sim$ ) real Bott manifolds  $M(A)$ ,  $M(B)$ , see [3] and [11], there are three operations on  $\mathcal{B}(n)$ , denoted by (Op1), (Op2) and (Op3), such that  $M(A) \sim M(B)$  if and only if the matrix  $A$  can be transformed into  $B$  through a sequence of the above operations, see [3, Part 3]. The operation (Op1) is a conjugation by a permutation matrix, (Op2) is a bijection  $\Phi_k: \mathcal{B}(n) \rightarrow \mathcal{B}(n)$

$$(8) \quad \Phi_k(A)_{*,j} := A_{*,j} + a_{kj}A_{*,k},$$

<sup>2</sup>Sometimes  $\mathcal{B}(n)$  is defined to be the set of strictly upper triangular binary matrices of size  $n$ .

for  $k, j \in \{1, 2, \dots, n\}$  such that  $\Phi_k \circ \Phi_k = 1_{\mathcal{B}(n)}$ . Finally (Op3) is, for distinct  $l, m \in \{1, 2, \dots, n\}$  and the matrix  $A$  with  $A_{*,l} = A_{*,m}$

$$(9) \quad \Phi^{l,m}(A)_{i,*} := \begin{cases} A_{l,*} + A_{m,*} & \text{if } i = m, \\ A_{i,*} & \text{otherwise.} \end{cases}$$

Here  $A_{*,j}$  denotes  $j$ -th column and  $A_{i,*}$  denotes  $i$ -th row of the matrix  $A$ .

Let us start to consider the relations between these two classes of flat manifolds. We start with an easy observation

$$\begin{aligned} \mathcal{RBM}(n) \cap \mathcal{GHW}(n) &= \{M(A) \mid \text{rank}_{\mathbb{Z}_2} A = n - 1\} \\ &= \{M(A) \mid a_{1,2}a_{2,3} \cdots a_{n-1,n} = 1\}. \end{aligned}$$

These manifolds are classified in [3, Example 3.2] and for  $n \geq 2$

$$(10) \quad \#(\mathcal{RBM}(n) \cap \mathcal{GHW}(n)) = 2^{(n-2)(n-3)/2}.$$

There exists the classification, see [17] and [3], of diffeomorphism classes of  $\mathcal{GHW}$  and  $\mathcal{RBM}$  manifolds in low dimensions. For  $\dim \leq 6$  we have the following table.

dim	number of $GHW$ manifolds		number of $RBM$ manifolds		number of $GHW \cap RBM$ manifolds
	total	oriented	total	oriented	total
1	0	0	1	1	0
2	1	0	2	1	0
3	3	1	4	2	1
4	12	0	12	3	2
5	123	2	54	8	8
6	2536	0	472	29	64

**Proposition 1.** *Let  $\Gamma_n$  be a group from Example 1 then  $\Gamma_n \in \mathcal{GHW} \cap \mathcal{RBM}$ .*

*Proof.* It is enough to see that the group  $(G, 0)\Gamma_n(G, 0)^{-1} = \Gamma(A)$ , where  $G = [g_{ij}]$ ,  $1 \leq i, j \leq n$ ,

$$g_{ij} := \begin{cases} 1 & \text{if } j = n - i + 1, \\ 0 & \text{otherwise,} \end{cases}$$

and  $A = [a_{ij}]$ ,  $1 \leq i, j \leq n$ , with

$$a_{ij} := \begin{cases} 1 & \text{if } j = i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

□

### 3. Existence of Spin and $\text{Spin}^{\mathbb{C}}$ structures on real Bott manifolds

In this section we shall give some condition for the existence of Spin and  $\text{Spin}^{\mathbb{C}}$  structures on real Bott manifolds. We use notations from the previous sections. There are a few ways to decide whether there exists a Spin structure on an oriented flat manifold  $M^n$ , see [7]. We start with the following. A closed oriented differential manifold  $N$  has such a structure if and only if the second Stiefel–Whitney class  $w_2(N) = 0$ . In the case of an oriented real Bott manifold  $M(A)$  we have the formula for  $w_2$ .

Recall, see [11], that for the Bott matrix  $A$

$$(11) \quad H^*(M(A); \mathbb{Z}_2) = \mathbb{Z}_2[x_1, x_2, \dots, x_n] \Big/ \left( x_j^2 = x_j \sum_{i=1}^n a_{i,j} x_i \mid j = 1, 2, \dots, n \right)$$

as graded rings. Moreover, from [12, (3.1) on p. 3] the  $k$ -th Stiefel–Whitney class

$$(12) \quad w_k(M(A)) = (B(p))^* \sigma_k(y_1, y_2, \dots, y_n) \in H^k(M(A); \mathbb{Z}_2),$$

where  $\sigma_k$  is the  $k$ -th elementary symmetric function,

$$p: \pi_1(M(A)) \rightarrow G \subset O(n)$$

a holonomy representation,  $B(p)$  is a map induced by  $p$  on the classification spaces and  $y_i := w_1(L_{i-1})$ , see (6). Hence,

$$(13) \quad w_2(M(A)) = \sum_{1 \leq i < j \leq n} y_i y_j \in H^2(M(A); \mathbb{Z}_2).$$

There exists a general condition, see [5, Theorem 3.3], for the calculation of the second Stiefel–Whitney for flat manifolds with  $(\mathbb{Z}_2)^k$  holonomy of diagonal type but we prefer the above explicit formula (13).<sup>3</sup> Its advantage follows from the knowledge of the cohomology ring (11) of real Bott manifolds.

An equivalent condition for the existence of a Spin structure is as follows. An oriented flat manifold  $M^n$  (a Bieberbach group  $\pi_1(M^n) = \Gamma$ ) has a Spin structure if and only if there exists a homomorphism  $\epsilon: \Gamma \rightarrow \text{Spin}(n)$  such that  $\lambda_n \epsilon = p$ . Here  $\lambda_n: \text{Spin}(n) \rightarrow SO(n)$  is the covering map, see [7]. We have a similar condition, under assumption  $H^2(M^n, \mathbb{R}) = 0$ , for the existence of  $\text{Spin}^{\mathbb{C}}$  structure, [7, Theorem 1]. In this case  $M^n$  (a Bieberbach group  $\Gamma$ ) has a  $\text{Spin}^{\mathbb{C}}$  structure if and only if there exists a homomorphism

$$(14) \quad \bar{\epsilon}: \Gamma \rightarrow \text{Spin}^{\mathbb{C}}(n)$$

such that  $\bar{\lambda}_n \bar{\epsilon} = p$ .  $\bar{\lambda}_n: \text{Spin}^{\mathbb{C}}(n) \rightarrow SO(n)$  is the homomorphism induced by  $\lambda_n$ , see [7]. We have the following easy observation. If there exists  $H \subset \Gamma$ , a subgroup of

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<sup>3</sup>We use it in Example 2.

finite index, such that the finite covering  $\tilde{M}^n$  with  $\pi_1(\tilde{M}^n) = H$  has no Spin ( $\text{Spin}^{\mathbb{C}}$ ) structure, then  $M^n$  has also no such structure. We shall prove.

**Theorem 1.** *Let  $A \in \mathcal{B}(n)$  be a matrix of an orientable real Bott manifold  $M(A)$ .*

I. *Let  $l \in \mathbb{N}$  be an odd number. If there exist  $1 \leq i < j \leq n$  and rows  $A_{i,*}, A_{j,*}$  such that*

$$(15) \quad \#\{m \mid a_{i,m} = a_{j,m} = 1\} = l$$

*and*

$$(16) \quad a_{i,j} = 0,$$

*then  $M(A)$  has no Spin structure. Moreover, if*

$$(17) \quad \#\{J \subset \{1, 2, \dots, n\} \mid \#J = 2, \sum_{j \in J} A_{*,j} = 0\} = 0,$$

*then  $M(A)$  has no  $\text{Spin}^{\mathbb{C}}$  structure.*

II. *If  $a_{ij} = 1$  and there exist  $1 \leq i < j \leq n$  and rows*

$$A_{i,*} = (0, \dots, 0, a_{i,i_1}, \dots, a_{i,i_{2k}}, 0, \dots, 0),$$

$$A_{j,*} = (0, \dots, 0, a_{j,i_{2k}+1}, \dots, a_{j,i_{2k}+2l}, 0, \dots, 0)$$

*such that  $a_{i,i_1} = a_{i,i_2} = \dots = a_{i,i_{2k}} = 1$ ,  $a_{i,m} = 0$  for  $m \notin \{i_1, i_2, \dots, i_{2k}\}$ ,  $a_{j,i_{2k}+1} = a_{j,i_{2k}+2} = \dots = a_{j,i_{2k}+2l} = 1$ ,  $a_{j,r} = 0$  for  $r \notin \{i_{2k} + 1, i_{2k} + 2, \dots, i_{2k} + 2l\}$  and  $l, k$  odd then  $M(A)$  has no Spin structure.*

**Proof.** From [11, Lemma 2.1] the manifold  $M(A)$  is orientable if and only if for any  $i = 1, 2, \dots, n$ ,

$$\sum_{k=i+1}^n a_{i,k} = 0 \pmod{2}.$$

Assume that  $\epsilon: \pi_1(M(A)) \rightarrow \text{Spin}(n)$  defines a Spin structure on  $M(A)$ . Let  $a_{i,i_1}, a_{i,i_2}, \dots, a_{i,i_{2m}}, a_{j,j_1}, a_{j,j_2}, \dots, a_{j,j_{2p}} = 1$  and let  $s_i, s_j$  be elements of  $\pi_1(M(A))$  which define rows  $i, j$  of  $A$ , see (7). Then

$$\epsilon(s_i) = \pm e_{i_1} e_{i_2} \cdots e_{i_{2m}}, \quad \epsilon(s_j) = \pm e_{j_1} e_{j_2} \cdots e_{j_{2p}}$$

and

$$\epsilon(s_i s_j) = \pm e_{k_1} e_{k_2} \cdots e_{k_{2r}}.$$

From (15)  $2r = 2m + 2p - 2l$ . Moreover  $\epsilon(s_i^2) = (-1)^m$ ,  $\epsilon(s_j^2) = (-1)^p$  and  $\epsilon((s_i s_j)^2) = (-1)^{m+p-l} = (-1)^{m+p+l}$ . Since from (16) (see also [11, Lemma 3.2])  $s_i s_j = s_j s_i$  we have  $\epsilon((s_i)^2) \epsilon((s_j)^2) = \epsilon((s_i s_j)^2)$ . Hence

$$(-1)^{m+p} = (-1)^{m+p+l}.$$

This is impossible since  $l$  is an odd number and we have a contradiction.

For the existence of the  $\text{Spin}^{\mathbb{C}}$  structure it is enough to observe that the condition (17) is equivalent to equation  $H^2(M(A), \mathbb{R}) = 0$ , see [3, formula (8.1)]. Hence, we can apply the formula (14). Let us assume that  $\bar{\epsilon}: \pi_1(M(A)) \rightarrow \text{Spin}^{\mathbb{C}}(n)$  defines a  $\text{Spin}^{\mathbb{C}}$  structure. Using the same arguments as above, see [7, Proposition 1], we obtain a contradiction. This finished the proof of I. For the proof II let us observe that  $s_i^2 = (s_i s_j)^2$ . Hence  $(-1)^k = \epsilon((s_i)^2) = \epsilon((s_i s_j)^2) = (-1)^{k+l} = 1$ . This is impossible.  $\square$

In the above theorem rows of number  $i$  and  $j$  correspond to generators  $s_i, s_j$  which define a finite index subgroup  $H \subset \pi_1(M(A))$ . It is a Bieberbach group with holonomy group  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ . We proved that  $H$  (if it exists) has no  $\text{Spin}$  ( $\text{Spin}^{\mathbb{C}}$ ) structure, (see the discussion before Theorem 1). In the next example we give the list of all 5-dimensional real Bott manifolds (with) without  $\text{Spin}$  ( $\text{Spin}^{\mathbb{C}}$ ) structure.

EXAMPLE 2. From [15] we have the list of all 5-dimensional oriented real Bott manifolds. There are 7 such manifolds without the torus. Here are their matrices:

$$\begin{aligned}
 A_4 &= \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, & A_{23} &= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
 A_{29} &= \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, & A_{37} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
 A_{40} &= \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, & A_{48} &= \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
 A_{49} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
 \end{aligned}$$

From the first part of Theorem 1 above, for  $i = 1, j = 2$  the manifold  $M(A_4)$  has no  $\text{Spin}^{\mathbb{C}}$  structure. For the same reasons (for  $i = 1, j = 2$ ) manifolds  $M(A_{40})$  and  $M(A_{48})$  have no  $\text{Spin}$  structures. The manifold  $M(A_{23})$  has no a  $\text{Spin}$  structure, because it satisfies



for  $i = 1, j = 3$  the second part of the Theorem 1. Since any flat oriented manifold with  $\mathbb{Z}_2$  holonomy has Spin structure, [10, Theorem 3.1] manifolds  $M(A_{29}), M(A_{49})$  have it. Our last example, the manifold  $M(A_{37})$  has Spin structure and we leave it as an exercise.

In all these cases it is possible to calculate the  $w_2$  with the help of (6), (13) and (11). In fact,  $w_2(M(A_4)) = (x_2)^2 + x_1x_3$ ,  $w_2(M(A_{23})) = x_1x_3$ ,  $w_2(M(A_{40})) = w_2(M(A_{48})) = x_1x_2$ . In all other cases  $w_2 = 0$ .

EXAMPLE 3. Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

be a family of Bott matrices, with  $* \in \mathbb{Z}_2$ . It is easy to check that the first two rows satisfy the condition of Theorem 1. Hence the oriented real Bott manifolds  $M(A)$  have no the Spin structure.

REMARK 1. In [1] on p.6 an example of the flat (real Bott) manifold  $M$  without Spin structure is considered. By an immediate calculation the Bott matrix of  $M$  is equal to

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

#### 4. Concluding Remarks

The tower (5) is an analogy of a Bott tower

$$W_n \rightarrow W_{n-1} \rightarrow \cdots \rightarrow W_1 = \mathbb{C}P^1 \rightarrow W_0 = \{\text{a point}\}$$

where  $W_i$  is a  $\mathbb{C}P^1$  bundle on  $W_{i-1}$  i.e.  $W_i = P(1 \oplus L_{i-1})$  and  $L_{i-1}$  is a holomorphic line bundle over  $W_{i-1}$ . As in (5)  $P(1 \oplus L_{i-1})$  is projectivisation of the trivial linear bundle and  $L_{i-1}$ . It was introduced by Grossberg and Karshon [8]. Note that  $W_n$  is a toric manifold. It means a normal algebraic variety over the complex numbers  $\mathbb{C}$  with an effective algebraic action of  $(\mathbb{C}^*)^n$  having an open dense orbit, where  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ . See [3] for the more complete bibliography.

There is an open problem: Is it true that two toric manifolds are diffeomorphic (or homeomorphic) if their cohomology rings with integer coefficients are isomorphic as graded rings? In some cases it has partial affirmative solutions. For the survey of those problems we send a reader to [4] and [11].

For real Bott manifolds the following is true.

**Theorem** ([11, Theorem 1.1]). *Two real Bott manifolds are diffeomorphic if and only if their cohomology rings with  $\mathbb{Z}_2$  coefficients are isomorphic as graded rings. Equivalently, they are cohomologically rigid.*

All of this suggests the following:

QUESTION. Are  $\mathcal{GHW}$ -manifolds cohomologically rigid?

The answer to the above question is positive for manifolds from  $\mathcal{GHW} \cap \mathcal{RBM}$ . It looks the most interesting for oriented GHW-manifolds. However, for  $n = 5$  there are two oriented Hantzsche–Wendt manifolds. From direct calculations with the help of a computer we know that they have different cohomology rings with  $\mathbb{Z}_2$  coefficients.

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